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WANDERING OUT TO INFINITY OF DIFFUSION PROCESSES

BY

AVNER FRIEDMAN[1]

ABSTRACT. Let $\xi(t)$ be a diffusion process in R^n, given by $d\xi = b(\xi)dt + \sigma(\xi)dw$. Conditions are given under which either $|\xi(t)| \to \infty$ as $t \to \infty$ with probability 1, or $\xi(t)$ visits any neighborhood at a sequence of times increasing to infinity, with probability 1. The results are obtained both in case (i) $\sigma(x)$ is nondegenerate, and (ii) $\sigma(x)$ is degenerate at a finite number of points and hypersurfaces.

Introduction. Let $w(t) = (w^1(t), \cdots, w^n(t))$ be an n-dimensional Brownian motion. It is well known [6, p. 236] that

(i) if $n \geq 3$ then, as $t \to \infty$, $w(t)$ wanders out to ∞ in the sense that $P\{|w(t)| \to \infty \text{ if } t \to \infty\} = 1$;

(ii) if $n = 2$ then $w(t)$ visits each disc at a sequence of times increasing to ∞.

The purpose of this paper is to establish this behavior for general diffusion processes; more specifically, for solutions $\xi(t)$ of stochastic differential systems. For simplicity, we shall take the coefficients to be time-independent. Thus, $\xi(t)$ is the solution of an n-dimensional system

$$d\xi(t) = b(\xi(t))dt + \sigma(\xi(t))dw(t), \tag{0.1}$$

where $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))$, $b(x) = (b_1(x), \cdots, b_n(x))$, and $\sigma(x)$ is $n \times n$ matrix $(\sigma_{ij}(x))$.

In §§1, 2 we consider the case where the matrix $\sigma(x)$ is nondegenerate for all $x \in R^n$. In §1, we prove a theorem of the form (i), and in §2 we prove a theorem of the form (ii). The conditions under which these theorems are proved are rather sharp. Moreover, they do not depend exclusively upon the dimension n. Thus, some diffusion processes wander out to ∞ even though $n = 2$, whereas others, with $n \geq 3$, visit any ball at a sequence of time increasing to ∞.

In §§3, 4 we consider the case where the matrix $\sigma(x)$ is degenerate.

Finally, in §5 we complement the results of §1 by deriving an estimate on the

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rate at which $|\xi(t)| \to \infty$ as $t \to \infty$; it is proved that

$$P_x\{|\xi(t)|/t^\theta \quad \infty \text{ as } t - \infty\} = 1, \quad \text{for any } 0 < \theta < \tfrac{1}{2}.$$

1. Nondegenerate diffusion: Wandering out to infinity. Set

$$a_{ij} = \frac{1}{2} \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk},$$

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}.$$

We shall assume

(A_1) For all $x \in R^n$,

$$\sum_{i=1}^{n} |b_i(x)| + \sum_{i,j=1}^{n} |\sigma_{ij}(x)| \leq C(1 + |x|) \qquad (C \text{ constant});$$

for any $R > 0$ there is a positive constant C_R such that

$$\sum_{i=1}^{n} |b_i(x) - b_i(y)| + \sum_{i,j=1}^{n} |\sigma_{ij}(x) - \sigma_{ij}(y)| \leq C_R |x - y|$$

if $|x| < R$, $|y| < R$.

(A_2) The matrix $(a_{ij}(x))$ is positive definite for each $x \in R^n$.

Let

$$A(x, \xi) = \frac{1}{|\xi|^2} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j,$$

$$B(x) = \sum_{i=1}^{n} a_{ii}(x), \qquad C(x, \xi) = \sum_{i=1}^{n} \xi_i b_i(x),$$

and set

$$S(x, \xi) = \frac{B(x) - A(x, \xi) + C(x, \xi)}{A(x, \xi)}, \qquad S(x) = S(x, x). \tag{1.1}$$

We shall need the assumption:

(A_3) There is a positive constant R_0 such that

$$S(x) \geq 1 + \epsilon(|x|) \quad \text{if } |x| \geq R_0, \tag{1.2}$$

where $\epsilon(r)$ is a continuous function satisfying

(1.3) $$\int_{R_0}^{\infty} \frac{1}{t} \exp\left[-\int_{R_0}^{t} \frac{\epsilon(s)}{s}\, ds\right] dt < \infty.$$

Notice that (1.3) holds for any of the functions $\epsilon(s) = d$, $\epsilon(s) = A/s$, $\epsilon(s) = B/(\log s)$ where d, A, B are positive constants and $B > 1$.

Theorem 1.1. *Let* (A_1)–(A_3) *hold. Then, for any* $x \in R^n$,

(1.4) $$P_x\{|\xi(t)| \to \infty \text{ if } t \to \infty\} = 1.$$

Proof. Let $\alpha > 0$. By (A_3), there is a continuous function $\theta(r)$ defined for $r \geq \alpha$ such that

(1.5) $$S(x) \geq \theta(|x|) \quad \text{if } |x| \geq \alpha,$$

(1.6) $$\theta(r) = 1 + \epsilon(r) \quad \text{if } r \geq R_0.$$

We shall construct a function $f(x) = F(r)$, where $r = |x|$, such that $Lf(x) \leq 0$ if $|x| \geq \alpha$. As easily verified,

(1.7) $$Lf(x) = A(x, x)F''(r) + (F'(r)/r)[B(x) - A(x, x) + C(x, x)].$$

Hence, if

(1.8) $$F'(r) \leq 0 \quad \text{for } r \geq \alpha,$$

(1.9) $$F''(r) + (\theta(r)/r)F'(r) = 0 \quad \text{for } r \geq \alpha,$$

then, by (1.5)–(1.7),

(1.10) $$Lf(x) \leq 0 \quad \text{if } |x| \geq \alpha.$$

Set

(1.11) $$I(r) = \int_{\alpha}^{r} \frac{\theta(s)}{s}\, ds.$$

Then a solution of (1.9) is given by

(1.12) $$F(r) = \int_{r}^{\infty} e^{-I(s)} ds;$$

the integral is convergent, by (1.3), (1.6). Notice that (1.8) is also satisfied. Hence (1.10) holds when F is given by (1.12).

By Itô's formula and (1.10), if $|x| > \alpha$ then

(1.13) $$E_x F(|\xi(\tau)|) - E_x F(|x|) = E_x \int_0^{\tau} Lf(\xi(s))ds \leq 0$$

where τ is any bounded stopping time such that $|\xi(s)| \geq \alpha$ if $0 \leq s \leq \tau$. Let $\beta > \alpha$, and let $\tau_{\alpha\beta}$ denote the exit time from the shell $\{y;\ \alpha < |y| < \beta\}$. Denote by $P_x(\alpha)$ the probability that $|\xi(\tau_{\alpha\beta})| = \alpha$ (given $\xi(0) = x$) and by $P_x(\beta)$ the probability that $|\xi(\tau_{\alpha\beta})| = \beta$ (given $\xi(0) = x$). Setting $\tau = T \wedge \tau_{\alpha\beta}$ in (1.13) and taking $T \to \infty$, we get (since $\tau_{\alpha\beta} < \infty$ a.s.; see Theorem 1 of [3]),

$$F(\alpha)P_x(\alpha) + F(\beta)P_x(\beta) \leq F(|x|).$$

Taking $\beta \to \infty$ and using the fact that $F(\beta) \to 0$ if $\beta \to \infty$, we get

$$\lim_{\beta\to\infty} P_x(\alpha) \leq \frac{F(|x|)}{F(\alpha)}. \tag{1.14}$$

Introduce the balls $B_\rho = \{y;\ |y| \leq \rho\}$ $(0 < \rho < \infty)$ and the event

$$\Omega(\alpha) = \{\xi(t) \text{ hits the ball } B_\alpha \text{ for some } t \geq 0\}. \tag{1.15}$$

Then we can write (1.14) in the form:

$$P_x(\Omega(\alpha)) \leq F(|x|)/F(\alpha). \tag{1.16}$$

Denote by t_R the hitting time of the boundary of the ball B_R by $\xi(t)$. By [3, Theorem 1], if $|x| < R$ then $E_x t_R < \infty$; consequently, $P_x\{t_R < \infty\} = 1$. Introduce the event

$$\Omega^*(\alpha) = \{\xi(t) \text{ hits the ball } B_\alpha \text{ at a sequence of times increasing to } \infty\}. \tag{1.17}$$

Thus $\Omega^*(\alpha)$ is a subset of $\Omega(\alpha)$.

Let $\alpha < |x| < R$. Since $P_x\{t_R < \infty\} = 1$,

$$P_x(\Omega^*(\alpha)) = P_x\{\xi(t + t_R) \text{ hits } B_\alpha \text{ at a sequence of times increasing to } \infty\}.$$

Using the strong Markov property, we get

$$P_x(\Omega^*(\alpha)) = E_x P_{\xi(t_R)}(\Omega^*(\alpha))$$

$$\leq E_x P_{\xi(t_R)}(\Omega(\alpha)) \leq EF(R)/F(\alpha) = F(R)/F(\alpha),$$

where (1.16) has been used. Taking $R \to \infty$, we get $P_x(\Omega^*(\alpha)) = 0$. This means that

$$P_x\left\{\varliminf_{t\to\infty} |\xi(t)| \geq \alpha\right\} = 1.$$

Since α is arbitrary, we get

$$P_x\left\{\varlimsup_{t\to\infty} |\xi(t)| = \infty\right\} = 1,$$

i.e., (1.4) holds.

We shall now replace the condition (A_3) by

(A_3') As $|x| \to \infty$,

$$a_{ij}(x) \to a_{ij}^0, \tag{1.18}$$

$$\sum_{i=1}^{n} x_i b_i(x) \to 0, \tag{1.19}$$

where the matrix (a_{ij}^0) has at least three positive eigenvalues.

Theorem 1.2. *Let* (A_1), (A_2), (A_3') *hold. Then, for any* $x \in R^n$, *the assertion* (1.4) *holds.*

Proof. We can perform a nonsingular transformation $x \to x'$ in R^n which takes (a_{ij}^0) into (a_{ij}^*), where $a_{ij}^* = 0$ if $i \neq j$, $a_{ii}^* = 1$ if $i = 1, 2, 3$ and $a_{ii}^* = 0$ or 1 if $4 \leq i \leq n$. In the new coordinates the condition (A_3) holds with $\epsilon(s) = d$ where d is any positive constant < 1. Now apply Theorem 1.1.

2. Nondegenerate diffusion: Visiting small neighborhoods. We shall replace the condition (A_3) by

(A_4) For any $z \in R^n$ there is a positive constant R_z such that

$$S(x, x - z) \leq 1 + \epsilon(|x - z|) \quad \text{if } |x - z| \geq R_z, \tag{2.1}$$

where $\epsilon(r)$ is a continuous function satisfying

$$\int_{R_*}^{\infty} \frac{1}{t} \exp\left[-\int_{R_*}^{t} \frac{\epsilon(s)}{s}\, ds\right] dt = \infty \quad \text{for some } R_* > 0. \tag{2.2}$$

For simplicity we have taken $\epsilon(r)$ to be independent of z; but the subsequent results are unaffected if $\epsilon(r)$ is allowed to depend on z.

Notice that the function $\epsilon(r) = 1/(\log r)$ satisfies (2.2).

Theorem 2.1. *Let* (A_1), (A_2), (A_4) *hold. Then, for any* $x \in R^n$ *and for any ball* $B_\alpha(z) = \{y;\ |y - z| \leq \alpha\}$, $\alpha > 0$,

(2.3) $P_x\{\xi(t)$ *hits* $B_\alpha(z)$ *at a sequence of times increasing to* $\infty\} = 1$.

Proof. We take, for simplicity, $z = 0$ and write $B_\alpha = B_\alpha(0)$. We shall first construct a function $f(x) = F(r)$ for $r = |x| \geq \alpha$ such that

(2.4) $$Lf(x) \geq 0 \quad \text{if } |x| \geq \alpha.$$

Let $\theta(r)$ be a continuous function such that

(2.5) $$S(x) \leq \theta(|x|) \qquad \text{if } |x| \geq \alpha,$$

(2.6) $$\theta(r) = 1 + \epsilon(r) \quad \text{if } r \geq R_0.$$

In view of (1.7), if $F(r)$ satisfies (1.8), (1.9) then (2.4) holds. With the definition (1.11), the function

(2.7) $$F(r) = -\int_\alpha^r e^{-I(s)} ds$$

satisfies both (1.9) and (1.8). In view of (2.6), (2.2),

(2.8) $$F(r) \to -\infty \quad \text{if } r \to \infty.$$

We shall now apply the first part of (1.13) to the present function $F(r)$. Making use of (2.4) and taking $\tau = \tau_{\alpha\beta} \wedge T$, we get, after letting $T \to \infty$,

(2.9) $$F(\alpha)P_x(\alpha) + F(\beta)P_x(\beta) - F(|x|) \geq 0.$$

Taking $\beta \to \infty$ in (2.9) and using (2.8), we conclude that $P_x(\beta) \to 0$ if $\beta \to \infty$. Hence, $P_x(\alpha) = 1 - P_x(\beta) \to 1$ if $\beta \to \infty$. This means that

(2.10) $$P_x(\Omega(\alpha)) = 1$$

where $\Omega(\alpha)$ is defined in (1.15).

For any $\rho > 0$, let $\partial B_\rho = \{y;\ |y| = \rho\}$. Let $\alpha < R_1 < R_2 < \cdots < R_m < \cdots$, $R_m \to \infty$ if $m \to \infty$. Introduce Markov times

τ_1 = first time $\xi(t)$ hits B_α;

σ_1 = first time $> \tau_1$ such that $\xi(t)$ hits ∂B_{R_1};

in general,

τ_m = first time $> \sigma_{m-1}$ such that $\xi(t)$ hits B_α;

σ_m = first time $> \tau_m$ such that $\xi(t)$ hits ∂B_{R_m}.

By (2.10), $\tau_1 < \infty$ a.s. Hence, by the strong Markov property,

$$P_x(\sigma_1 < \infty) = P_x(\tau_1 < \infty, \sigma_1 < \infty) = E_x E_{\xi(\tau_1)} \chi_{\sigma_1 - \tau_1 < \infty}.$$

By Theorem 1 of [3], $P_z(\sigma_1 - \tau_1 < \infty) = 1$ if $|z| < R_1$. Since $|\xi(\tau_1)| = \alpha < R_1$ a.s., we get $P_x(\sigma_1 < \infty) = 1$.

We now proceed by induction. Assuming that $\tau_m < \infty$ a.s., we get

$$P_x(\sigma_m < \infty) = P_x(\tau_m < \infty, \sigma_m < \infty) = E_x E_{\xi(\tau_m)} \chi_{\sigma_m - \tau_m < \infty} = 1$$

since $P_z(\sigma_m - \tau_m < \infty) = 1$ if $|z| < R_m$ (by [3]). Next

$$P_x(\tau_{m+1} < \infty) = P_x(\sigma_m < \infty, \tau_{m+1} < \infty) = E_x E_{\xi(\sigma_m)} \chi_{\tau_{m+1} - \sigma_m < \infty} = 1$$

since, by (2.10), $P_z(\tau_{m+1} - \sigma_m < \infty) = 1$ for any $z \in R^n$, $|z| \geq \alpha$.

We have thus proved, by induction, that $\tau_m < \infty$ a.s. for all m.

Now, at each time $t = \tau_m$, $\xi(t)$ hits B_α. Further, since $|\xi(\sigma_m)| = R_m \longrightarrow \infty$ as $m \longrightarrow \infty$, $\lim_{m\to\infty} \sigma_m = \infty$; hence also $\lim_{m\to\infty} \tau_m = \infty$. This completes the proof of (2.3) (in case $z = 0$).

We shall now replace the condition (A_4) by

(A_4') As $|x| \longrightarrow \infty$,

$$a_{ij}(x) - a_{ij}^0 = o(1/\log|x|), \tag{2.11}$$

$$\sum |b_i(x)| = o(1/|x|\log|x|), \tag{2.12}$$

and the matrix (a_{ij}^0) has precisely two positive eigenvalues.

Theorem 2.2. *Let* (A_1), (A_2), (A_4') *hold. Then, for any* $x \in R^n$, *the assertion* (2.3) *holds for any* $z \in R^n$, $\alpha > 0$.

Proof. We perform an orthogonal transformation $x \longrightarrow x'$ that takes (a_{ij}^0) into a new matrix (a_{ij}^*) with $a_{ij}^* = 0$ if $i \neq j$ or if $i = j \geq 3$, and $a_{ii}^* = 1$ if $i = 1, 2$. In the new coordinates, the condition (A_4) is satisfied with $\epsilon(r) = 1/(\log r)$. Now apply Theorem 2.1.

Remark. Suppose (A_4') is replaced by

$$a_{ij}(x) \longrightarrow a_{ij}^0 \quad \text{as } |x| \longrightarrow \infty,$$

$$\sum |b_i(x)| = o(1/|x|) \quad \text{as } |x| \longrightarrow \infty$$

where the matrix (a_{ij}^0) has precisely one positive eigenvalue. Then the assertion of Theorem 2.1 remains valid, with the same proof; here $\epsilon(r) = d$ where d is any positive constant < 1.

Example. Consider the case where $n \geq 2$, $b_i \equiv 0$ and σ is such that

$$\tfrac{1}{2}\sigma\sigma^* = (a_{ij}), \qquad a_{ij} = \delta_{ij} + (g(r)/r^2)x_i x_j \qquad (r = |x|);$$

$g(r)$ is a Lipschitz continuous function vanishing near $r = 0$, and

$$\mu \leq g(r) \leq M \quad \text{where } \mu > -1,\ M < \infty \ (\mu, M \text{ constants}).$$

The eigenvalues of $(a_{ij}(x))$ are 1 (with multiplicity $n-1$) and $1+g$. Hence $(a_{ij}(x))$ is positive definite for all $x \in R^n$. Clearly,

$$S(x) - 1 = (n-2-g(r))/(1+g(r)) \equiv \epsilon(r).$$

Hence, if $g(r)$ is such that $\epsilon(r) = A/(\log r)$ for some $A > 1$ (and all large r) then the assertion of Theorem 1.1 holds. If $g(r)$ is such that $\epsilon(r) = 1/(\log r)$, then (A_4) holds with $\epsilon(s) = 1/(\log s) + C/s$ for some positive constant C; consequently the assertion of Theorem 2.1 holds. This example shows that the conditions (A_3), (A_4) made in Theorems 1.1, 2.1 are rather sharp.

This example also shows that the behavior asserted in Theorems 1.1 and 2.1 does not depend exclusively on the dimension n. In fact, given any $\epsilon(r)$ which converges to 0 as $r \to \infty$, take $g = (n-2+\epsilon)/(r+\epsilon)$ (for all large r) in the above example. Then the behavior of $\xi(t)$ does not depend on n; if $\epsilon(r) \geq A/(\log r)$ for some $A > 1$ then $\xi(t)$ wanders out to ∞ (i.e., (2.3) holds), whereas if $\epsilon(r) \leq 1/(\log r)$ then $\xi(t)$ visits any neighborhood at a sequence of times increasing to ∞.

Meyers and Serrin [7] have introduced the conditions (A_3), (A_4) in studying the existence and uniqueness of bounded solutions u for the exterior Dirichlet problem. When (A_3) holds, they prescribe u at ∞, but when (A_4) holds they only require that u be bounded near ∞. The above example is taken from [7].

3. Degenerate diffusion: Wandering out to infinity. We shall now allow the matrix $(a_{ij}(x))$ to degenerate on a compact subset of R^n. This set will consist of a finite number of points $G_1 = \{z_1\}, \cdots, G_{k_0} = \{z_{k_0}\}$ and of a finite number of hypersurfaces $\partial G_{k_0+1}, \cdots, \partial G_{k_0+k_1}$, where G_{k_0+i} is a closed bounded domain with boundary ∂G_{k_0+i}. It is assumed that $G_i \cap G_j = \emptyset$ if $i \neq j$. Let

$$k = k_0 + k_1, \qquad G = \bigcup_{i=1}^{k} G_i, \qquad \hat{G} = R^n \backslash G.$$

Denote by $\rho_i(x)$ the distance from x to G_i when $x \notin \operatorname{int} G_i$, and let

$$\epsilon_0 = \min_{i \neq j} \operatorname{dist}(G_i, G_j).$$

We shall assume

(B_1) ∂G_h $(k_0 + 1 \leq h \leq k)$ is in C^3. If $1 \leq h \leq k_0$ then $b_i(z_h) = 0$, $\sigma_{ij}(z_h) = 0$ for $1 \leq i, j \leq n$. If $k_0 + 1 \leq h \leq k$ then

(3.1) $$\sum_{i,j=1}^{n} a_{ij}\nu_i\nu_j = 0 \quad \text{on } \partial G_b,$$

(3.2) $$(b, \nu) + \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 \rho_b}{\partial x_i \partial x_j} \geq 0 \quad \text{on } \partial G_b$$

where ν is the outward normal to ∂G_b. Finally, the matrix $(a_{ij}(x))$ is nondegenerate for any $x \in \hat{G}$.

If (A_1), (B_1) hold then, by Friedman and Pinsky [5, Theorem 1.1], for any $x \in \hat{G}$,

$$P_x\{\xi(t) \text{ hits } G \text{ at some time } t \geq 0\} = 0.$$

By [5], there exists a continuous function $R(x)$ in R^n, which has the following properties:

(i) $R(x)$ is in $C^2(\hat{G})$;

(ii) $R(x) > 0$ in $\hat{G}$;

(iii) $R(x) = \rho_b(x)$ if $\rho_b(x) \leq \epsilon_0$; $R(x) > \epsilon_0$ if $\min_b \rho_b(x) > \epsilon_0$;

(iv) $R(x) = |x|$ if $|x|$ is sufficiently large;

(v) $\sum_{i,j=1}^{n} a_{ij}(x)\, \partial^2 R/\partial x_i \partial x_j < 0$ if $x \in \hat{G}$, $\nabla_x R(x) = 0$;

there are precisely $k - 1$ points x in $\hat{G}$ with $\nabla_x R(x) = 0$.

Let

$$\mathfrak{A} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j},$$

$$\mathfrak{B} = \sum_{i=1}^{n} b_i(x)\frac{\partial R}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 R}{\partial x_i \partial x_j},$$

$$Q = (1/R)(\mathfrak{B} - \mathfrak{A}/R).$$

Then, if $g(x) = \Phi(R(x))$,

(3.3) $$Lg(x) = \mathfrak{A}[\Phi''(R) + \Phi'(R)/R] + RQ\Phi'(R).$$

Clearly

$$S(x) = 1 + R^2(x)Q(x)/\mathfrak{A}(x) \quad \text{if } |x| \text{ is sufficiently large.}$$

As proved in [5], if $\overline{\lim}\, Q(x) < 0$ as $R(x) \to 0$ and as $R(x) \to \infty$, then

$$P_x\left\{\min_{1\leq b\leq k} [\rho_b(\xi(t))] \to 0 \text{ if } t \to \infty\right\} = 1 \quad \text{if } x \in \hat{G};$$

Thus, in this case, $\xi(t)$ neither wanders out to ∞ nor visits any neighborhood in $\hat{G}$ at a sequence of times increasing to ∞. Hence, in order to obtain the type of behavior asserted in §§1, 2, we shall have to change the conditions on $Q(x)$.

As will be shown, in order to generalize the results of §§1, 2, to the present case where (B_1) holds, we do not need to change the conditions (A_3), (A_4) near ∞. We only need to impose a condition on $Q(x)$ near $R(x) = 0$. This condition is

(B_2) For some $0 < \delta_0 < \epsilon_0$ there is a continuous function $\epsilon(r)$, defined for $0 < r \leq \delta_0$, such that

$$Q(x) \geq (\mathcal{A}(x)/R^2(x))\epsilon(R(x)) \quad \text{if } 0 < R(x) \leq \delta_0, \tag{3.4}$$

and

$$\int_r^{\delta_0} \frac{1}{s} \exp\left[\int_s^{\delta_0} \frac{\epsilon(t)}{t}\,dt\right] ds \to \infty \quad \text{if } r \to 0. \tag{3.5}$$

We can take, for example, $\epsilon(s) = -1/[\log(1/s)]$.

Remark. The condition: $\overline{\lim}\, Q(x) < 0$ as $R(x) \to 0$ is a "stability condition", meaning that G attracts $\xi(t)$ near the boundary. The condition (3. 4) can be interpreted as a weak "repelling condition".

Theorem 3.1. *Let* (A_1), (B_1), (B_2) *and* (A_3) *hold. Then, for any* $x \in \hat{G}$,

$$P_x\{|\xi(t)| \to \infty \quad \text{if } t \to \infty\} = 1. \tag{3.6}$$

Let G be contained in a ball $B_{R_*} = \{y;\ |y| < R_*\}$.

We shall first prove a lemma.

Lemma 3.2. *Let* (A_1), (B_1), (B_2) *hold and let* $\beta > R_*$. *Then, for any* $x \in \hat{G} \cap B_\beta$,

$$P_x\{\xi(t) \text{ hits the set } \partial B_\beta \text{ for some } t \geq 0\} = 1. \tag{3.7}$$

Here B_β is the ball $\{y;\ |y| \leq \beta\}$ and ∂B_β is its boundary.

Proof. We first construct a function $g(x) = \Phi(R(x))$ for $x \in \hat{G} \cap B_\beta$, such that

$$Lg(x) \leq 0 \quad \text{if } x \in \hat{G} \cap B_\beta. \tag{3.8}$$

Denote by $z_1, \cdots, z_{k-1}$ the points in $\hat{G}$ where $\nabla_x R(x) = 0$. By slightly modifying the proof of Lemma 2.1 in [5], we obtain a modified function $R(x)$ for which the points $z_1, \cdots, z_{k-1}$ lie outside the ball B_β. We shall work, in the present proof, with this modified function $R(x)$; it coincides with the original $R(x)$ in the ϵ_0-neighborhood of G.

We claim that there is a continuous function $\theta(r)$ satisfying

$$1 + R^2(x)Q(x)/\mathfrak{A}(x) \geq \theta(R(x)) \quad \text{if } x \in \hat{G} \cap B_\beta, \tag{3.9}$$

$$\theta(r) = 1 + \epsilon(r) \quad \text{if } 0 < r \leq \delta_0. \tag{3.10}$$

Indeed, since $\mathfrak{A}(x) \neq 0$ if $x \in \hat{G}$, $\nabla_x R(x) \neq 0$, the left-hand side of (3.9) is a bounded function if $x \in \hat{G} \cap B_\beta$, $\min_b \rho_b(x) \geq \delta_0$. Using the assumption (3.4), the existence of $\theta(r)$ [satisfying (3.9), (3.10)] follows.

Let $\Phi(r)$ be a solution of

$$\Phi''(r) + \theta(r)\Phi'(r)/r = 0 \quad \text{if } 0 < r < r_0, \tag{3.11}$$

$$\Phi'(r) \leq 0 \quad \text{if } 0 < r < r_0 \tag{3.12}$$

where $r_0 = \max_{|x| \leq \beta} R(x)$. Then, upon using (3.3), (3.9) we conclude that $g(x) = \Phi(R(x))$ satisfies (3.8).

A solution of (3.11), (3.12) is given by

$$\Phi(r) = \int_r^{\delta_0} \exp\left[\int_s^{\delta_0} \frac{\theta(t)}{t}\, dt\right] ds. \tag{3.13}$$

In view of (3.5),

$$\Phi(r) \longrightarrow \infty \quad \text{if } r \longrightarrow 0. \tag{3.14}$$

By Itô's formula and (3.8)

$$E_x\Phi(|\xi(\tau)|) - E_x\Phi(|x|) = E_x \int_0^\tau Lg(\xi(s))\,ds \leq 0, \tag{3.15}$$

where τ is any bounded stopping time such that $\xi(s) \in \hat{G} \cap B_\beta$ if $0 \leq s \leq \tau$. Denote by G_ϵ $(\epsilon > 0)$ the closed ϵ-neighborhood of G, and denote by $\sigma_{\epsilon\beta}$ the hitting time of the set $G_\epsilon \cup \partial B_\beta$. By [3, Theorem 1], $P_x(\sigma_{\epsilon\beta} < \infty) = 1$ if $x \notin G_\epsilon$, $x \in B_\beta$. Denote by $P_x(\epsilon)$ the probability that $\xi(\sigma_{\epsilon\beta}) \in G_\epsilon$ (given $\xi(0) = x$), and by $P_x(\beta)$ the probability that $\xi(\sigma_{\epsilon\beta}) \in \partial B_\beta$ (given $\xi(0) = x$). Substituting $\tau = \sigma_{\epsilon\beta} \wedge T$ in (3.15), and taking $T \longrightarrow \infty$, we get

$$\Phi(\epsilon)P_x(\epsilon) + \Phi(\beta)P_x(\beta) \leq \Phi(|x|).$$

Taking $\epsilon \longrightarrow 0$ and using (3.14), we deduce that $P_x(\epsilon) \longrightarrow 0$ if $\epsilon \longrightarrow 0$. Hence $P_x(\beta) \longrightarrow 1$ if $\epsilon \longrightarrow 0$. But this implies the assertion of the lemma.

Let $R^* < \alpha < R$ and denote by t_R the hitting time of the ball B_R. By

Lemma 3.2, $P_x(t_R < \infty) = 1$ if $x \in \hat{G}$. Hence, by the strong Markov property (cf. the proof of Theorem 1.1),

$$P_x'(\Omega^*(\alpha)) = EP_{\xi(t_R)}(\Omega^*(\alpha)) \leq EP_{\xi(t_R)}(\Omega(\alpha)),$$

where the notation (1.15), (1.17) is used.

Now, in the domain $\{y;\ |y| > R^*\}$ the matrix $(a_{ij}(y))$ is nondegenerate. Since the condition (A_3) holds, the estimate (1.14) remains valid for $x = \xi(t_R)$. Hence

$$P_x(\Omega^*(\alpha)) \leq F(R)/F(\alpha) \qquad (F(r) \to 0 \text{ if } r \to \infty).$$

We can now complete the proof, as in the case of Theorem 1.1, by taking $R \to \infty$ and noting that α can be arbitrarily large.

Theorem 3.3. *Let* (A_1), (B_1), (B_2) *and* (A_3') *hold. Then for any* $x \in \hat{G}$ *the assertion* (3.6) *holds.*

Proof. Proceeding as in the proof of Theorem 3.1, it remains to establish the estimate (1.14). We now perform a nonsingular transformation as in the proof of Theorem 1.2.

4. Degenerate diffusion: Visiting small neighborhoods.

Theorem 4.1. *Let* (A_1), (B_1), (B_2) *and* (A_4) *hold. Then, for any ball* $B_\alpha(z) = \{y;\ |y - z| \leq \alpha\}$, $\alpha > 0$, *lying entirely in* $\hat{G}$, *the assertion* (2.3) *holds.*

Proof. For simplicity we take $z = 0$. Let R be any positive number such that $R > \alpha$ and such that G is contained in the interior of the ball B_R. We shall prove: if $x \in \partial B_R$ then

$$P_x\{\xi(t) \text{ hits the ball } B_\alpha \text{ for some } t \geq 0\} = 1. \tag{4.1}$$

Let $\delta > 0$ be sufficiently small so that $\delta < \epsilon_0$, the closed δ-neighborhood G_δ of G lies in the interior of B_R, and $G_\delta \cap B_\alpha = \emptyset$.

We shall construct a function $f(x) = F(R(x))$ in $R^n \setminus G_\delta$ such that

$$Lf(x) \geq 0 \quad \text{if } x \in R^n \setminus G_\delta, \tag{4.2}$$

$$F(r) \to -\infty \quad \text{if } r \to \infty. \tag{4.3}$$

Notice that at the points z_m $(m = 1, \cdots, k-1)$ where $\nabla_x R(x) = 0$, $\mathfrak{A} = 0$ and, therefore, by the property (v) of $R(x)$, $Q < 0$. Hence, $Q(x) < 0$ in a neighborhood of each point z_m. It follows that there is a continuous function $\theta(r)$, $\delta \leq r < \infty$, such that

$$(4.4) \qquad 1 + R^2 Q/\mathcal{Q} \leq \theta(R) \quad \text{if } x \in R^n \backslash G_\delta.$$

In view of (A_4), we can choose $\theta(r)$ so that, for all r sufficiently large, $\theta(r) = 1 + \epsilon(r)$ where $\epsilon(r)$ satisfies (2.2). If we now define $F(r)$, for $r \geq \delta$, by

$$F(r) = -\int_\delta^r e^{-I(s)} ds, \qquad I(s) = \int_\delta^s \frac{\theta(t)}{t} dt$$

then $F'(r) \leq 0$, and (4.2), (4.3) hold. Arguing as in the proof of Theorem 2.1 (following (2.8)) with the present function $f(x) = F(R(x))$ and with the set $\{y;\ |y| \leq \alpha\}$ replaced by G_δ, we conclude that for any $x \in R^n \backslash G_\delta$,

$$(4.5) \qquad P_x\{\xi(t) \text{ hits the set } G_\delta \text{ for some } t \geq 0\} = 1.$$

Let $\beta > R$ and denote by $\hat{G}_{\alpha\beta\delta}$ the domain bounded by ∂G_δ, ∂B_α, ∂B_β. Denote by $\tau_{\alpha\beta}$ the exit time from this domain. By [3, Theorem 1], $P_x(\tau_{\alpha\beta} < \infty) = 1$ if $x \in \hat{G}_{\alpha\beta\delta}$. Using the strong Markov property we get, for any $x \in \partial B_R$,

$$\begin{aligned}
&P_x\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad = P_x\{\xi(t + \tau_{\alpha\beta}) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad = E_x P_{\xi(\tau_{\alpha\beta})}\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad \geq P_x\{\xi(\tau_{\alpha\beta}) \in \partial B_\alpha\} + P_x\{\xi(\tau_{\alpha\beta}) \in \partial G_\delta\} \inf_{y \in \partial G_\delta} P_y\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\}.
\end{aligned}$$

Denote by $\sigma_{\alpha R}$ the hitting time of $B_\alpha \cup \partial B_R$. By Lemma 3.2, if $y \in \partial G_\delta$, then $P_y(\sigma_{\alpha R} < \infty) = 1$. Hence, by the strong Markov property,

$$\begin{aligned}
&P_y\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad = P_y\{\xi(t + \sigma_{\alpha R}) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad = E_y P_{\xi(\sigma_{\alpha R})}\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\} \\
&\quad = P_y\{\xi(\sigma_{\alpha R}) \in \partial B_\alpha\} \\
&\qquad + (1 - P_y\{\xi(\sigma_{\alpha R}) \in \partial B_\alpha\}) \inf_{x \in \partial B_R} P_x\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\}.
\end{aligned}$$

Combining this with the previous inequality, and setting

$$\begin{aligned}
P_x(\alpha) &= P_x\{\xi(t) \text{ hits } B_\alpha \text{ for some } t \geq 0\}, \\
\gamma_{\alpha\beta}(x) &= P_x\{\xi(\tau_{\alpha R}) \in \partial B_\alpha\}, \\
\gamma_\beta(x) &= P_x\{\xi(\tau_{\alpha R}) \in \partial G_\delta\}, \\
\mu(y) &= P_y\{\xi(\sigma_{\alpha R}) \in \partial B_\alpha\},
\end{aligned}$$

we arrive at the inequality,

$$(4.6)\qquad P_x(\alpha) \geq \gamma_{\alpha\beta}(x) + \gamma_\beta(x) \inf_{y \in \partial G_\delta} \left\{ \mu(y) + [1 - \mu(y)] \inf_{z \in \partial B_R} P_z(\alpha) \right\}.$$

Note that $\gamma_{\alpha\beta}(x)$ is the solution $u(x)$ of the Dirichlet problem:

$$Lu = 0 \quad \text{in } \hat{G}_{\alpha\beta\delta},$$

$$u = 1 \quad \text{on } \partial B_\alpha,$$

$$u = 0 \quad \text{on } \partial G_\delta \cup \partial B_\beta.$$

Hence, by the strong maximum principle, $\gamma_{\alpha\beta}(x)$ is positive on ∂B_R. Further, $\gamma_{\alpha\beta}(x) \nearrow$ if $\beta \nearrow$. Similar assertions are true for $\gamma_\beta(x)$. Since $\gamma_{\alpha\beta}(x) + \gamma_\beta(x) \leq 1$, we conclude that

$$(4.7)\qquad \gamma_\beta(x) \leq \theta < 1 \qquad (x \in \partial B_R)$$

where θ is a constant *independent* of β.

Notice that (4.5) implies that

$$(4.8)\qquad \gamma_{\alpha\beta}(x) + \gamma_\beta(x) \longrightarrow 1 \quad \text{if } \beta \longrightarrow \infty \qquad (x \in \partial B_R).$$

Let $P_\alpha = \inf_{z \in \partial B_R} P_z(\alpha)$. Let η be a positive number, and choose x_0 in ∂B_R so that $P_\alpha \geq P_{x_0}(\alpha) - \eta$. Let y_0 be a point in ∂G_δ such that

$$\inf_{y \in \partial G_\delta} \{\mu(y) + [1 - \mu(y)]P_\alpha\} \geq \{\mu(y_0) + [1 - \mu(y_0)]\}P_\alpha - \eta.$$

Applying (4.6) with $x = x_0$, we get

$$P_\alpha \geq \gamma_{\alpha\beta}(x_0) + \gamma_\beta(x_0)\{\mu(y_0) + [1 - \mu(y_0)]P_\alpha\} - 2\eta.$$

Hence

$$P_\alpha\{1 - \gamma_\beta(x_0)[1 - \mu(y_0)]\} \geq \gamma_{\alpha\beta}(x_0) + \gamma_\beta(x_0)\mu(y_0) - 2\eta.$$

Taking β sufficiently large and using (4.8) with $x = x_0$, we get

$$P_\alpha\{1 - \gamma_\beta(x_0)[1 - \mu(y_0)]\} \geq \{1 - \gamma_\beta(x_0)[1 - \mu(y_0)]\} - 3\eta.$$

Denote the expression in braces by λ_β. From (4.7) it follows that $\lambda_\beta \geq 1 - \theta > 0$. Hence

$$P_\alpha \geq 1 - 3\eta/\lambda_\beta \geq 1 - 3\eta/(1 - \theta).$$

Since η is arbitrary, $P_\alpha = 1$. This implies that $P_\alpha(x) = 1$ for all $x \in \partial G_R$, i.e., (4.1) holds.

Having proved (4.1), we can now easily complete the proof of Theorem 4.1 by the argument given in the proof of Theorem 2.1 (from (2.10) on). Instead of (2.10) we use (4.1), and instead of Theorem 1 of [3] we use Lemma 3.2.

Remark 1. Theorem 4.1 remains true if the condition (A_4) is replaced by (A_4'), or by the conditions in the remark following Theorem 2.2.

Remark 2. When the $\sigma_{ij}(x)$ are linear homogeneous functions, the matrix $(a_{ij}(x))$ degenerates at the origin and (possibly) along rays initiating at the origin. In case the $b_i(x)$ are also linear homogeneous functions the behavior of $\xi(t)$ was studied in detail in [4]. The function $Q(x)$ is now homogeneous. If $Q(y) < 0$ for all y, $|y| = 1$, then $P_x\{\xi(t) \to 0 \text{ if } t \to \infty\} = 1$ for all x. If $Q(y) > 0$ for all y, $|y| = 1$, then $P_x\{|\xi(t)| \to \infty \text{ if } t \to \infty\} = 1$ for all $x \neq 0$.

5. The rate of wandering out to infinity. In this section we return to the situation of Theorem 1.1. We shall assume

(A_5) $a_{ij}(x)$, $b_i(x)$ are bounded functions in R^n, the $a_{ij}(x)$ are uniformly Hölder continuous in R^n, and, for all $x \in R^n$, $\xi \in R^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2 \qquad (\alpha_0 \text{ positive constant}).$$

We shall also assume that the function $\epsilon(r)$ occurring in the condition (A_3) satisfies, for some r_0 sufficiently large,

$$\epsilon(r) = d \quad \text{if } r \geq r_0 \qquad (d \text{ positive constant}). \tag{5.1}$$

Theorem 5.1. *Let (A_1), (A_5) hold and let (A_3) hold with $\epsilon(r)$ satisfying (5.1). Then, for any $0 < \theta < 1/2$, $x \in R^n$,*

$$P_x\{|\xi(t)|/t^\theta \to \infty \text{ if } t \to \infty\} = 1. \tag{5.2}$$

For $\xi(t)$ a Brownian motion $w(t)$, this result was proved by Dvoretsky and Erdös [1]. As in [1], the function t^θ occurring in (5.2) can actually be replaced by $t^{1/2}g(t)\lambda(t)$ where $g(t)$, $\lambda(t)$ are positive, monotone decreasing, $\lambda(t) \to 0$ if $t \to \infty$, and

$$\sum_{n=1}^{\infty} [g(2^m)]^{d_0} < \infty \quad \text{for some } d_0 \leq d,\ d_0 < n.$$

More recently, Theorem 5.1, under more restrictive assumptions, was proved by Friedman [3]. The proof in [3] exploits the result of Dvoretsky-Erdös together with estimates on $E_x|\xi(t)-\sigma_0 w(t)|^2$, where $\sigma_0 = \lim \sigma(x)$ as $|x| \to \infty$. The present proof is entirely different from that in [3]; it is modeled after the proof in [1] for $\xi(t) = w(t)$.

We shall first prove two lemmas.

Lemma 5.2. *Let* (A_1), (A_5) *and* (A_3), (5.1) *hold. Then there exists a positive constant C such that for any* $\alpha > r_0$, $x \in R^n$,

$$P_x\{|\xi(t)| \leq \alpha \text{ for some } t \geq 0\} \leq C(\alpha/|x|)^d. \tag{5.3}$$

Proof. Let $R > \alpha$. Consider the Dirichlet problem:

$$Lu_R(x) = 0 \quad \text{if } \alpha < |x| < R,$$

$$u_R = 1 \quad \text{if } |x| = \alpha,\ u_R = 0 \text{ if } |x| = R.$$

Let $f(x) = F(r)$ $(r = |x|)$ be the function constructed in the proof of Theorem 1.1. Then $v(x) = F(r)/F(\alpha)$ $(r = |x|)$ satisfies

$$Lv \leq 0 \quad \text{if } \alpha < |x| < R,$$

$$v = 1 \quad \text{if } |x| = \alpha, \qquad v \geq 0, \quad \text{if } |x| = R.$$

Hence, by the maximum principle,

$$0 \leq u_R(x) \leq v(x). \tag{5.4}$$

By a standard argument (cf. [7]), there exists a sequence $R = R_m \nearrow \infty$ for which the $u_R(x)$ converge point-wise to a solution u of

$$Lu = 0 \quad \text{if } |x| > \alpha, \qquad u = 1 \quad \text{if } |x| = 1. \tag{5.5}$$

From (5.4) we get

$$0 \leq u(x) \leq v(x) = \frac{\int_r^\infty \exp[-I(s)]\,ds}{\int_\alpha^\infty \exp[-I(s)]\,ds}.$$

Using (5.1) to estimate the right-hand side, we obtain,

$$0 \leq u(x) \leq C\alpha^d/r^d \tag{5.6}$$

where C is a constant independent of α, x.

Now, by Itô's formula, if $\alpha < |x| < R$,

$$u_R(x) = P_x\{\xi(t) \text{ hits } \partial B_\alpha \text{ before it hits } \partial B_R\}.$$

Hence, if $|x| > \alpha$,

(5.7) $$u(x) = P_x\{\xi(t) \text{ hits } \partial B_\alpha \text{ for some } t \geq 0\}.$$

Now, if $|x| \leq \alpha$ then the assertion (5.3) is trivially true (with $C = 1$). If, on the other hand, $|x| > \alpha$ then the assertion (5.3) follows from (5.7), (5.6).

Lemma 5.3. *Let* (A_1), (A_5) *and* (A_3), (5.1) *hold, with* $d < n$. *Then there is a positive constant* C' *such that, if* $\alpha > r_0$, $|x| < \alpha/4$, $T > 0$,

(5.8) $$P_x\{|\xi(t)| \leq \alpha \text{ for some } t \geq T\} \leq C'(\alpha/T^{1/2})^d.$$

Proof. By the Markov property and Lemma 5.2,

(5.9)
$$\begin{aligned}
&P_x\{|\xi(t)| \leq \alpha \text{ for some } t \geq T\}\\
&\quad = E_x P_{\xi(T)}\{|\xi(t)| \leq \alpha \text{ for some } t \geq 0\}\\
&\quad = P_x\{|\xi(T)| \leq \alpha\} + E_x\{\chi_{|\xi(T)|>\alpha}[C\alpha^d/|\xi(T)|^d]\}\\
&\quad \equiv I + J.
\end{aligned}$$

Denote by $K(t, x, y)$ the fundamental solution of the parabolic operator $\partial/\partial t - L$. By [2],

(5.10) $$0 \leq K(t, x, y) \leq (M/t^{n/2}) \exp[-\mu|x - y|^2/t]$$

where M, μ are positive constants.

We can write

$$I = \int_{|y|\leq\alpha} K(T, x, y)\, dy,$$

$$J = C\alpha^d \int_{|y|>\alpha} \frac{1}{|y|^d} K(T, x, y)\, dy.$$

We shall subsequently denote various positive constants by the same symbol C. Substituting $|y - x| = \rho\sqrt{T}$ in the integral I and noting that

$$\rho\sqrt{T} = |y - x| \leq 2\alpha \quad (\text{since } |y| \leq \alpha,\ |x| < \alpha),$$

we get

$$I \leq C \int_{\rho\sqrt{T}\leq 2\alpha} \rho^{n-1} e^{-\mu\rho^2}\, d\rho \leq C\left(\frac{\alpha}{\sqrt{T}}\right)^n.$$

Substituting $|y - x| = \rho\sqrt{T}$ in the integral J and noting that

$$\rho\sqrt{T} = |y - x| \geq \alpha/2 \quad (\text{since } |y| \geq \alpha,\ |x| \leq \alpha/2),$$

$$|y| \geq |y - x| - |x| \geq \rho\sqrt{T}/2 \quad (\text{since } |x| \leq \alpha/4 \leq |y - x|/2),$$

we get

$$J \leq C\alpha^d \int_{\rho\sqrt{T}>\alpha/2} \frac{\rho^{n-1}}{(\rho\sqrt{T})^d} e^{-\mu\rho^2} d\rho$$

$$\leq C\left(\frac{\alpha}{\sqrt{T}}\right)^d \int_{\rho>0} \rho^{n-1-d} e^{-\mu\rho^2} d\rho \leq C\left(\frac{\alpha}{\sqrt{T}}\right)^d$$

since $n-1-d>-1$. Substituting the estimates for I, J into (5.9), the assertion (5.8) follows.

Proof of Theorem 5.1. Without loss of generality we may assume that $d<n$. We apply Lemma 5.3 with $T=2^m$, $\alpha=2^{(m+1)\theta}$ where m is a positive integer such that $|x|<2^{(m+1)\theta}/4$. We get

$$P_x\{|\xi(t)|\leq t^\theta \text{ for some } t,\ 2^m\leq t\leq 2^{m+1}\}$$

$$\leq P_x\{|\xi(t)|\leq 2^{(m+1)\theta} \text{ for some } t\geq 2^m\}$$

$$\leq C[2^{(m+1)\theta}/2^{m/2}]^d \leq C2^{m(\theta-1/2)d}.$$

Since $\Sigma\, 2^{m(\theta-1/2)d}<\infty$, the Borel-Cantelli lemma implies that, with probability 1, the sequence of events

$$\{|\xi(t)|\leq t^\theta \text{ for some } t,\ 2^m\leq t\leq 2^{m+1}\}$$

(where $\xi(0)=x$) occurs only finitely often. Hence

$$P_x\{|\xi(t)|>t^\theta \text{ for all } t \text{ sufficiently large}\}=1.$$

Since this is true also for θ replaced by any θ', $\theta<\theta'<1/2$, the assertion of the theorem follows.

Corollary 5.4. *Let* (A_1), (A_5) *and* (A_3') *hold. Then, for any* $0<\theta<1/2$, $x\in R^n$, *the assertion* (5.2) *holds.*

Indeed, perform a nonsingular transformation as in the proof of Theorem 1.2. In the new coordinates the conditions (A_3), (5.1) hold.

Remark. Consider the example at the end of §2. If $n=2$ and $g=-1/(1+1/d)$ $(d>0)$, then the assertion of Theorem 5.1 holds. Thus, even when $n=2$, a diffusion process $\xi(t)$ may wander out to ∞ at a rate $\geq t^\theta$, for any $0<\theta<1/2$.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201